The bispectral property of a *q*-deformation of the Schur polynomials and the *q*-KdV hierarchy

1997 J. Phys. A: Math. Gen. 30 7217

(http://iopscience.iop.org/0305-4470/30/20/021)

J. Phys. A: Math. Gen. **30** (1997) 7217–7227. Printed in the UK PII: S0305-4470(97)85554-0

The bispectral property of a q-deformation of the Schur polynomials and the q-KdV hierarchy

Luc Haine† and Plamen Iliev‡

Department of Mathematics, Université Catholique de Louvain, 1348 Louvain-la-Neuve, Belgium

Received 2 May 1997

Abstract. We show that appropriate q-analogues of the Schur polynomials provide rational solutions of a q-deformation of the Nth KdV hierarchy. This allows us to construct explicit examples of bispectral commutative rings of q-difference operators.

1. Introduction

Recently, Frenkel [8] has introduced q-deformations of the KdV hierarchy and related soliton equations. His motivation was to elucidate the q-version of the, by now well known, connection between the classical W-algebra of $sl(N)$ and the bi-Hamiltonian structure of the Nth Korteweg–de Vries (KdV) hierarchy, see also [9]. Incidentally he raised the interesting open problem of finding explicit solutions of the q-deformed hierarchies.

In [11] Grünbaum and Haine have unearthed another potential area of application of q-deformations of soliton equations in the context of an extension of the classical orthogonal polynomials. Following Andrews and Askey [2], a family of orthogonal polynomials is called classical when they are eigenfunctions of a second-order q-difference operator, in addition to fulfilling the three-term recursion relation which is satisfied by any family of orthogonal polynomials. Classical orthogonal polynomials thus provide one of the earliest sources of 'bispectral situations'. In [11] all doubly infinite tridiagonal matrices were determined for which some families of eigenfunctions are also eigenfunctions of a second-order q-difference operator. The solution is described in terms of an arbitrary solution of a q-analogue of Gauss' hypergeometric equation depending on five free parameters and extends the four-dimensional family of solutions given by the well known Askey–Wilson polynomials [3]. Some further examples involving higher order recursion relations and q-difference equations were obtained in [10, 12] by application of the (matrix) Darboux transformation. In addition some evidence was presented that this 'difference, q-difference' bispectral problem is intimately related with some q-deformation of the Toda lattice hierarchy and its Virasoro symmetries, although the precise connection remains to be worked out.

The aim of this paper is to initiate the study of a (perhaps simpler) 'q-difference, q-difference' version of the bispectral problem and to connect it with a q-deformation of the Nth KdV hierarchy. Precisely, we show that q-analogues of the Schur polynomials solve the

† Research Associate of the Belgian National Fund for Scientific Research. E-mail address: haine@agel.ucl.ac.be
‡ E-mail address: iliev@agel.ucl.ac.be

q-deformed Nth KdV hierarchy (for an appropriate choice of N) and that the corresponding wave functions are eigenfunctions of a commutative ring of q-difference operators in the spectral parameter. This extends to the case $q \neq 1$ results of [1, 15, 7, 17, 18]. As in the case $q = 1$, a suitable adaptation of the Darboux transformation will play a crucial role in establishing these results. For $q = 1$, as was first shown in [7], it is well known that the Schur polynomials which solve the KdV hierarchy (corresponding to $N = 2$) provide the complete list of all rank-1 bispectral Schrödinger operators. However, for $N > 2$, as shown by Wilson [17], these solutions form a rather small subset among all rank-1 bispectral commutative rings of differential operators. The full set of solutions turns out to be parametrized by the so called adelic Grassmannian. We defer to a further work, a thorough study of a q-analogue of Wilson's adelic Grassmannian.

2. A q-deformation of the Nth KdV hierarchy

The q-derivative operator D_q acting on functions of x is defined by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}. \tag{2.1}$$

We shall also use the operator

$$(Df)(x) = f(qx) \tag{2.2}$$

and its powers

$$(D^k f)(x) = f(q^k x) \qquad k \in \mathbb{Z}. \tag{2.3}$$

Our definition of the q-deformed Nth KdV hierarchy will be slightly different from the one given in [8]. It has the advantage that it will obviously converge to the standard Nth KdV hierarchy when $q \to 1$. We consider q-difference operators of order N of the form

$$\begin{aligned} L = D_q^N + (q-1)x\left(\sum_{j=0}^{N-2}(-1)^j (q-1)^j x^j u_{j+2}(x)\right)D_q^{N-1} + u_2(x)D_q^{N-2} + \cdots \\ + u_{N-1}(x)D_q + u_N(x) \end{aligned} \tag{2.4}$$

or, equivalently, in D notation:

$$L = \frac{1}{q^{\frac{N(N-1)}{2}}(q-1)^N x^N}\left(D^N - v_1(x)D^{N-1} + \cdots + (-1)^{N-1}v_{N-1}(x)D + (-1)^N q^{\frac{N(N-1)}{2}}\right). \tag{2.5}$$

The Nth q-deformed KdV hierarchy is defined by the Lax equations

$$\frac{\partial L}{\partial t_j} = [(L^{j/N})_+, L] \qquad j = 1, 2, 3, \ldots \tag{2.6}$$

where $(L^{j/N})_+$ denotes the positive part (including D^0) of the 'pseudo-difference' operator $L^{j/N}$.

When $q = 1$, the Nth root of L is a pseudo-differential operator with coefficients in the ring $\mathbb{C}[\mathrm{d}^k/\mathrm{d}x^k u_i]_{i=2,\ldots,N;k\geqslant 0}$. When $q \neq 1$, one must be careful with the definition of $L^{1/N}$. To explain this, let us try to find

$$Q := L^{1/N} = \frac{1}{(q-1)x}(D + a_0(x) + a_1(x)D^{-1} + a_2(x)D^{-2} + \cdots)$$

so that $Q^N = L$. A short computation gives that

$$Q^N = \frac{1}{q^{\frac{N(N-1)}{2}}(q-1)^N x^N}\left[D^N + \left(\sum_{j=1}^{N} q^{j-1}D^{N-j}a_0\right)D^{N-1} + \cdots \right.$$
$$\left. + \left(\sum_{j=1}^{N} q^{k(j-1)}D^{N-j}a_{k-1} + \text{terms involving } a_0, a_1, \ldots, a_{k-2}\right)D^{N-k} + \cdots \right].$$

Thus $a_0(x), a_1(x), a_2(x), \ldots$ belong to a ring $R_{q,N}$ which can be defined as follows. Let $R^{(0)}_{q,N} = \mathbb{C}[v_i(q^k x)]_{i=1,\ldots,N-1;k\in\mathbb{Z}}$. Adjoin to $R^{(0)}_{q,N}$ all the solutions of

$$(D^{N-1} + qD^{N-2} + q^2D^{N-3} + \cdots + q^{N-1})a_0(x) = r(x) \tag{2.7}$$

for all $r(x) \in R^{(0)}_{q,N}$. This gives us a ring $R^{(1)}_{q,N}$. Then we adjoin to $R^{(1)}_{q,N}$ all the solutions of $(D^{N-1} + q^2D^{N-2} + q^4D^{N-3} + \cdots + q^{2(N-1)})a_1(x) = r(x)$, for all $r(x) \in R^{(1)}_{q,N}$, etc. The inductive limit of the rings $R^{(i)}_{q,N}$, which are obtained in this way, is the ring $R_{q,N}$. For instance the above equations can always be uniquely (formally) solved if we take the coefficients of $Lv_i(x)$ to be formal Laurent series in x. We refer the reader to [8] for details as well as for a description of the bi-Hamiltonian structure of the deformed hierarchy.

One crucial difference between the q-deformed hierarchy (2.6) and the standard one is that the first q-KdV flow is *no longer* a translation in x. It is given by

$$\frac{\partial L}{\partial t_1} = \left[D_q + \frac{a_0(x)+1}{(q-1)x}, L\right] \tag{2.8}$$

where $a_0(x)$ satisfies (2.7) with $r(x) = -v_1(x)$.

3. q-analogues of the Schur polynomials solving q-KdV

We shall now produce true (non-formal) solutions of (2.6) by means of the Darboux transformation applied to D_q^N.

The q-exponential function is defined by

$$\exp_q(x) = \sum_{k=0}^{\infty} \frac{(1-q)^k x^k}{(q;q)_k} \tag{3.1}$$

where $(a;q)_k$ denotes the standard q-shifted factorial

$$(a;q)_0 = 1 \qquad (a;q)_k = (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{k-1}) \qquad k \geqslant 1.$$

We introduce the elementary q-Schur polynomials $S_k(x;t_1,t_2,\ldots)$ by the generating function

$$\sum_{k\in\mathbb{Z}} S_k(x;t_1,t_2,\ldots)z^k = \exp_q(xz)\exp\left(\sum_{k=1}^{\infty} t_k z^k\right). \tag{3.2}$$

Thus

$$S_k = 0 \qquad \text{for } k < 0 \qquad S_0 = 1 \qquad S_1 = x + t_1$$
$$S_2 = \frac{q-1}{q^2-1}x^2 + t_1 x + \frac{t_1^2}{2} + t_2, \text{ etc.}$$

To each partition $\lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0\}$ formed with a non-increasing finite sequence of positive integers, we associate q-Schur polynomials $S_\lambda(x;t_1,t_2,\ldots)$ via the usual $n \times n$ determinant

$$S_\lambda(x;t_1,t_2,\ldots) = \det(S_{\lambda_i+j-i}(x;t_1,t_2,\ldots)). \tag{3.3}$$

For instance

$$S_{2,1}(x;t_1,t_2,t_3) = q\frac{q-1}{q^3-1}x^3 + t_1x^2 + t_1^2x + \frac{t_1^3}{3} - t_3. \tag{3.4}$$

By definition, when $q \to 1$, these polynomials depend only on $x + t_1$ and we can put $x = 0$, which agrees with the standard definition of the Schur polynomials. Denoting for a moment the standard Schur polynomials by $S_\lambda(t_1, t_2, \ldots)$ and their deformed version by $S_{\lambda,q}(x; t_1, t_2, \ldots)$, since

$$\exp_q(xz) = \exp\left(\sum_{k=1}^{\infty}\frac{(1-q)^k}{k(1-q^k)}x^kz^k\right) \tag{3.5}$$

we obtain immediately that

$$S_{\lambda,q}(x;t_1,t_2,t_3,\ldots) = S_\lambda\left(t_1+x, t_2+\frac{(1-q)^2}{2(1-q^2)}x^2, t_3+\frac{(1-q)^3}{3(1-q^3)}x^3,\ldots\right) \tag{3.6}$$

an observation that we shall exploit soon !

We define the q-Wronskian of functions $f_1, \ldots, f_n$ of x by

$$W_q(f_1,\ldots,f_n) = \det(D_q^{i-1}f_j)_{1\leqslant i,j\leqslant n}. \tag{3.7}$$

When $f_1, \ldots, f_n$ are also functions of t_1 we shall use the notation

$$W_{t_1}(f_1,\ldots,f_n) = \det\left(\left(\frac{\partial}{\partial t_1}\right)^{i-1}f_j\right)_{1\leqslant i,j\leqslant n} \tag{3.8}$$

to denote the usual Wronskian built with $\partial/\partial t_1$. It is clear from the definition (3.2) of the elementary Schur polynomials that

$$D_qS_k = \frac{\partial}{\partial t_1}S_k = S_{k-1} \tag{3.9}$$

and thus, for any partition $\lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0\}$, we have that

$$S_\lambda = W_q(S_{\ell_1}, S_{\ell_2}, \ldots, S_{\ell_n}) = W_{t_1}(S_{\ell_1}, S_{\ell_2}, \ldots, S_{\ell_n}) \tag{3.10}$$

with

$$\ell_1 = \lambda_n < \ell_2 = \lambda_{n-1} + 1 < \ell_3 = \lambda_{n-2} + 2 < \cdots < \ell_n = \lambda_1 + n - 1. \tag{3.11}$$

Let $\{k_1, \ldots, k_r\} = \{0, 1, 2, \ldots, \ell_n\}\backslash\{\ell_1, \ell_2, \ldots, \ell_n\}$, with $r = \ell_n + 1 - n$. Since the functions $S_0, S_1, S_2, \ldots, S_{\ell_n}$ span the kernel of $D_q^{\ell_n+1}$, we can factorize

$$D_q^{\ell_n+1} = QP \tag{3.12}$$

as a product of two monic q-difference operators $P = D_q^n +$ (lower) and $Q = D_q^r +$ (lower), with

$$Pf = \frac{W_q(S_{\ell_1}, S_{\ell_2}, \ldots, S_{\ell_n}, f)}{W_q(S_{\ell_1}, S_{\ell_2}, \ldots, S_{\ell_n})} \tag{3.13}$$

and

$$Qf = \frac{W_q(P(S_{k_1}), P(S_{k_2}), \ldots, P(S_{k_r}), f)}{W_q(P(S_{k_1}), P(S_{k_2}), \ldots, P(S_{k_r}))} \tag{3.14}$$

where f denotes an arbitrary function of x. In the following, we shall often use the notation

$$\tau(x,t) := W_q(S_{\ell_1}, S_{\ell_2}, \ldots, S_{\ell_n}) \tag{3.15}$$

with $t = (t_1, t_2, \ldots)$, anticipating the result that this Wronskian is indeed a 'tau-function' of the q-deformed Nth KdV hierarchy, for an appropriate choice of N. We can now state

Theorem 3.1. For any partition $\lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0\}$, let $N = \lambda_1 + n$ and let P, Q be the q-difference operators defined in (3.13) and (3.14) so that $D_q^N = QP$. Then, the operator

$$L = PQ \tag{3.16}$$

solves the q-deformed Nth KdV hierarchy (2.6).

Proof. Writing $LP = PD_q^N$ in 'D-notation' (2.3), one sees immediately that the coefficient of D^0 in L has to be as in (2.5) or, equivalently, that L has the form given in (2.4). Since $D_q^N \exp_q(xz) = z^N \exp_q(xz)$, the 'wave function'

$$\Psi = \frac{1}{z^n} P\left(\exp_q(xz)\exp\left(\sum_{k=1}^{\infty} t_k z^k\right)\right) \tag{3.17}$$

obviously satisfies

$$L\Psi = z^N \Psi. \tag{3.18}$$

Let us denote by $\Psi_{q=1}(t_1, t_2, \ldots)$ the wave function of the standard Nth KdV hierarchy corresponding to the usual Schur polynomial $S_\lambda(t_1, t_2, \ldots)$ with $q = 1$. It is given by the same formula as in (3.17) with $x = 0$, except that in the definition (3.13) of P, q is replaced by t_1 and f is a function of t_1, with W_{t_1} the usual Wronskian (3.8). This remark combined with (3.5), (3.6), (3.9) and the definition (3.17) of Ψ shows that

$$\Psi = \Psi_{q=1}\left(t_1 + x, t_2 + \frac{(1-q)^2}{2(1-q^2)}x^2, t_3 + \frac{(1-q)^3}{3(1-q^3)}x^3, \ldots\right).$$

Thus, by the classical result that the Schur polynomials with $q = 1$ are tau-functions of the KP hierarchy [15] (see also [16]), there exist differential operators $A_j(\partial/\partial t_1)$ in $\partial/\partial t_1$ so that

$$\frac{\partial \Psi}{\partial t_j} = A_j(\partial/\partial t_1)\Psi \qquad j = 2, 3, \ldots. \tag{3.19}$$

If we could establish that

$$\frac{\partial \Psi}{\partial t_1} = \left(D_q + \frac{\partial}{\partial t_1}\log\frac{\tau(qx,t)}{\tau(x,t)}\right)\Psi \tag{3.20}$$

with $\tau(x,t)$ as in (3.15), we could re-express the $\partial/\partial t_1$-derivatives in (3.19) in terms of D_q-derivatives, proving thus the existence of q-difference operators $B_j(D_q)$ so that

$$\frac{\partial \Psi}{\partial t_j} = B_j(D_q)\Psi \qquad j = 2, 3, \ldots. \tag{3.21}$$

It then follows immediately from (3.18), (3.20) and (3.21), using standard arguments, that L solves the q-deformed Nth KdV hierarchy (2.6).

It remains to establish (3.20). Substituting (3.2) into the definition (3.17) of Ψ gives

$$\Psi = \sum_{k=0}^{\infty} z^{k-n}\frac{W_q(S_{\ell_1}, S_{\ell_2}, \ldots, S_{\ell_n}, S_k)}{W_q(S_{\ell_1}, S_{\ell_2}, \ldots, S_{\ell_n})} \tag{3.22}$$

with $\ell_1, \ell_2, \ldots, \ell_n$ as in (3.11). Let us denote by $W_{\lambda,k}(x,t)$ the Wronskian $W_q(S_{\ell_1}, S_{\ell_2}, \ldots, S_{\ell_n}, S_k) = W_{t_1}(S_{\ell_1}, S_{\ell_2}, \ldots, S_{\ell_n}, S_k)$. Recalling the definition of $\tau(x,t)$ in (3.15), after plugging (3.22) into (3.20), the validity of this identity amounts to checking that

$$W_q(\tau(x,t), W_{\lambda,k}(x,t)) = W_{t_1}(\tau(qx,t), W_{\lambda,k}(x,t)) \tag{3.23}$$

with W_q and W_{t_1} as in (3.7) and (3.8). One can think of this identity as an identity between two differential operators in $\partial/\partial t_1$ of order $n+1$ acting on the polynomials S_k. Thus, if we can show that the coefficients of $(\partial/\partial t_1)^{n+1}$ and $(\partial/\partial t_1)^n$ are the same on both sides of (3.23), since the identity is automatically satisfied for $k = \ell_1, \ell_2, \ldots, \ell_n$, it will be identically satisfied. One computes easily that both the left-hand side and the right-hand side of (3.23), when expanded as differential operators in $\partial/\partial t_1$ acting on S_k, are given by

$$\tau(qx,t)\tau(x,t)\left(\frac{\partial}{\partial t_1}\right)^{n+1} S_k - \tau(x,t)\frac{\partial}{\partial t_1}\tau(qx,t)\left(\frac{\partial}{\partial t_1}\right)^{n} S_k + \text{(lower order terms)}$$

which establishes (3.23) and concludes the proof of the theorem. □

As in the case $q = 1$, for special choices of the partition λ, it may happen that L in (3.16) is a power of some q-difference operator. For instance, the well known 'staircase choice' $\lambda = \{\lambda_1 = n > \lambda_2 = n-1 > \lambda_3 = n-2 > \cdots > \lambda_n = 1\}$, corresponding to $\ell_1 = 1 < \ell_2 = 3 < \ell_3 = 5 < \cdots < \ell_n = 2n-1$ in (3.11), leads to an operator L such that $L = \tilde{L}^n$, with

$$\tilde{L} = D_q^2 + (q-1)xu(x,t)D_q + u(x,t) \tag{3.24}$$

a solution of the q-analogue of the KdV hierarchy itself, corresponding to $N = 2$. Moreover, the celebrated formula expressing $u(x,t)$ in terms of the tau-function admits the following q-analogue

$$u(x,t) = D_q \frac{\partial}{\partial t_1} \log \tau(x,t)\tau(qx,t) \tag{3.25}$$

with $\tau(x,t) = S_{n,n-1,n-2,\ldots,1}(x; t_1, t_2, \ldots)$ and D_q the standard q-derivative operator introduced in (2.1). To see this, observe that since

$$D_q^2 S_{\ell_k} = D_q^2 S_{2k-1} = S_{2k-3} = S_{\ell_{k-1}} \qquad 1 \leqslant k \leqslant n$$

by the definition of P (3.13), we have that $\ker P \subset \ker P D_q^2$ and thus

$$P D_q^2 = \tilde{L} P \tag{3.26}$$

with $\tilde{L}$ a second-order q-difference operator as in (3.24). Hence,

$$\tilde{L}^n = P D_q^{2n} P^{-1} = P(QP)P^{-1} = PQ = L.$$

To check formula (3.25), we equate the coefficients of D_q^{n+1} in (3.26), which gives

$$b(x) = (q-1)xu(x) + b(q^2 x) \tag{3.27}$$

with $b(x)$ the coefficient of D_q^{n-1} in P. Now, from the definition of P (3.13) and by using (3.9), (3.10) and (3.15), one easily shows that

$$\begin{aligned} b(x) &= -\frac{\partial}{\partial t_1} W_{t_1}(S_{\ell_1}, \ldots, S_{\ell_n}) / W_q(S_{\ell_1}, \ldots, S_{\ell_n}) \\ &= -\frac{\partial}{\partial t_1} \log \tau(x,t) \end{aligned}$$

which, using (3.27), gives (3.25).

4. Bispectral property of the q-Schur polynomials

Our proof of the bispectral property of the q-Schur polynomials will be based on a q-version of a very useful lemma due to Reach [14]. This lemma was used in [18] to prove that the classical Schur polynomials are bispectral. Compared with the more recent developments on the bispectral problem in [4, 5, 13], based on Wilson's idea of a bispectral involution [17], this lemma has the advantage of producing *explicit* formulae for a whole commutative ring of bispectral operators. At the end of the section, we shall connect this method with the more recent idea of the bispectral involution.

Lemma 4.1. Let $f_0, f_1, \ldots, f_{n+1}$ be functions of x. Define

$$F(x) = \sum_{k=1}^{n+1} (-1)^{n+1+k} f_k(x) \int f_0(x) W_q(f_1, \ldots, \hat{f}_k, \ldots, f_{n+1}) \, \mathrm{d}_q x. \tag{4.1}$$

Then

$$W_q(f_1, \ldots, f_n, F) = \theta(x) W_q(f_1, \ldots, f_{n+1}) \tag{4.2}$$

with

$$\theta(x) = \left(\int f_0(x) W_q(f_1, \ldots, f_n) \, \mathrm{d}_q x \right)_{|xq}. \tag{4.3}$$

Here $\int \mathrm{d}_q x$ denotes the standard q-integral.

Proof. Expanding with respect to the last row the identity

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_{n+1} \\ D_q f_1 & D_q f_2 & \cdots & D_q f_{n+1} \\ \vdots & \vdots & & \vdots \\ D_q^{n-1} f_1 & D_q^{n-1} f_2 & \cdots & D_q^{n-1} f_{n+1} \\ f_1(q^s x) & f_2(q^s x) & \cdots & f_{n+1}(q^s x) \end{vmatrix} = 0$$

for $s = 0, 1, \ldots, n-1$, gives

$$\sum_{k=1}^{n+1} (-1)^{n+1+k} f_k(q^s x) W_q(f_1, \ldots, \hat{f}_k, \ldots, f_{n+1}) = 0. \tag{4.4}$$

We now compute $D_q F, D_q^2 F, \ldots$. We have

$$D_q F = \sum_{k=1}^{n+1} (-1)^{n+1+k} f_k(qx) f_0(x) W_q(f_1, \ldots, \hat{f}_k, \ldots, f_{n+1}) + \sum_{k=1}^{n+1} (-1)^{n+1+k} D_q f_k \int f_0(x) W_q(f_1, \ldots, \hat{f}_k, \ldots, f_{n+1}) \, \mathrm{d}_q x.$$

The first term is zero by (4.4) with $s = 1$. Continuing the process inductively and taking appropriate linear combinations of the identities (4.4), we get that

$$D_q^j F = \sum_{k=1}^{n+1} (-1)^{n+1+k} D_q^j f_k \int f_0(x) W_q(f_1, \ldots, \hat{f}_k, \ldots, f_{n+1}) \, \mathrm{d}_q x \quad \text{for } j = 1, 2, \ldots, n-1. \tag{4.5}$$

This gives then that

$$D_q^n F = \sum_{k=1}^{n+1} (-1)^{n+1+k} (D_q^{n-1} f_k)(qx) f_0(x) W_q(f_1, \ldots, \hat{f}_k, \ldots, f_{n+1})$$
$$+ \sum_{k=1}^{n+1} (-1)^{n+1+k} D_q^n f_k \int f_0(x) W_q(f_1, \ldots, \hat{f}_k, \ldots, f_{n+1})\, \mathrm{d}_q x. \tag{4.6}$$

Note that now (and this is the main difference with the case $q = 1$) the first term is non-zero, but it would be zero if instead of $(D_q^{n-1} f_k)(qx)$ we had $(D_q^{n-1} f_k)(x)$. Thus we can rewrite $D_q^n F$ as

$$\begin{aligned} D_q^n F &= (q-1)x \sum_{k=1}^{n+1} (-1)^{n+1+k} (D_q^n f_k)(x) f_0(x) W_q(f_1, \ldots, \hat{f}_k, \ldots, f_{n+1}) \\ &\quad + \text{same second term as in (4.6)} \\ &= (q-1)x f_0(x) W_q(f_1, \ldots, f_{n+1}) \\ &\quad + \sum_{k=1}^{n+1} (-1)^{n+1+k} D_q^n f_k \int f_0(x) W_q(f_1, \ldots, \hat{f}_k, \ldots, f_{n+1})\, \mathrm{d}_q x. \end{aligned} \tag{4.7}$$

Putting (4.1), (4.5) and (4.7) into $W_q(f_1, \ldots, f_n, F)$, most of the terms disappear by column elimination and we obtain

$$\begin{aligned} W_q(f_1, \ldots, f_n, F) &= W_q(f_1, \ldots, f_{n+1}) \bigg[\int f_0(x) W_q(f_1, \ldots, f_n)\, \mathrm{d}_q x \\ &\quad + (q-1)x f_0(x) W_q(f_1, \ldots, f_n) \bigg] \\ &= \theta(x) W_q(f_1, \ldots, f_{n+1}) \end{aligned}$$

with $\theta(x)$ as in (4.3), which proves the lemma. □

The next theorem expresses the bispectral property of the q-analogues of the Schur polynomials.

Theorem 4.2. Let $\lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0\}$ be a partition and let $\tau(x, t) = S_\lambda(x; t_1, t_2, \ldots)$ be the associated q-Schur polynomial. Then the corresponding solution L of the q-deformed Nth KdV hierarchy built in theorem 3.1 is bispectral. Precisely, the function

$$\tilde{\Psi} = P\left(\frac{\exp_q(xz)}{z^n}\right) = \exp\left(-\sum_{k=1}^{\infty} t_k z^k\right)\Psi \tag{4.8}$$

with P and Ψ as in (3.13) and (3.17), satisfies

$$L\tilde{\Psi} = z^N \tilde{\Psi}$$

and, for any polynomial $\theta(x)$ such that $D_q\theta(x)$ is divisible by $\tau(qx, t)$, there exists a q-difference operator in z, $B(z, D_{q,z})$ independent of x such that

$$B(z, D_{q,z})\tilde{\Psi} = \theta(x)\tilde{\Psi}$$

with $D_{q,z}$ the standard q-derivative operator acting on functions of z defined by $(D_{q,z} f)(z) = (f(qz) - f(z))/(q-1)z$.

Proof. By q-integration by parts, for any polynomial $r(x)$, we have

$$\int r(x)\exp_q(xz)\,\mathrm{d}_q x = \sum_{k=0}^{\infty}\frac{(-1)^k}{q^{\frac{k(k+1)}{2}}z^{k+1}}(D_q^k r)\left(\frac{x}{q^{k+1}}\right)\exp_q(xz). \tag{4.9}$$

Now apply lemma 4.1 with f_0 an arbitrary polynomial in x, $f_1 = S_{\ell_1}, f_2 = S_{\ell_2}, \ldots, f_n = S_{\ell_n}, f_{n+1} = \exp_q(xz)/z^n$, with $\ell_1, \ell_2, \ldots, \ell_n$ as in (3.11). Using (4.9) we have that $F(x)$ can be expressed as

$$F(x) = R(x)\exp_q(xz)$$

with $R(x)$ some polynomial in x with coefficients depending rationally on z and polynomially on $t = (t_1, t_2, \ldots)$. Thus, rewriting $x^r \exp_q(xz)$ as $D_{q,z}^r \exp_q(xz)$, we get that

$$F(x) = R(D_{q,z})z^n\left(\frac{\exp_q(xz)}{z^n}\right) = B(z, D_{q,z})\left(\frac{\exp_q(xz)}{z^n}\right) \tag{4.10}$$

where, for notational convenience, we omit the explicit dependence in $t = (t_1, t_2, \ldots)$ of the coefficients of B, which we think of as free parameters. Putting (4.10) into (4.2) and using the definition of $\tilde{\Psi}$ (4.8) and (4.3), we obtain

$$B(z, D_{q,z})\tilde{\Psi} = \theta(x)\tilde{\Psi}$$

with

$$\theta(x) = \left(\int f_0(x)\tau(x,t)\,\mathrm{d}_q x\right)_{|xq} \tag{4.11}$$

from which it follows that $\theta(x)$ can be any polynomial in x such that $D_q\theta(x)$ is divisible by $\tau(qx, t)$. This concludes the proof of the theorem. □

By the same argument as in the case $q = 1$ (see for instance [6]) one shows that Sato's formula [15] is still valid:

$$\tilde{\Psi} = \frac{\tau(x, t_1 - 1/z, t_2 - 1/2z^2, t_3 - 1/3z^3, \ldots)}{\tau(x, t_1, t_2, t_3, \ldots)}\exp_q(xz).$$

From this formula and the definition of $\tilde{\Psi}$ (4.8), since $\tau(x, t)$ is polynomial in x, we deduce

$$\tilde{\Psi} = \frac{1}{z^n}P(\exp_q(xz)) = \frac{1}{\tau(x,t)}P_b(z, D_{q,z})\exp_q(xz) \tag{4.12}$$

with

$$P_b(z, D_{q,z}) = \tau(D_{q,z}, t_1 - 1/z, t_2 - 1/2z^2, t_3 - 1/3z^3, \ldots).$$

Thus any bispectral operator $B(z, D_{q,z})$ obtained via theorem 4.2 satisfies

$$(P_b^{-1}BP_b)\exp_q(xz) = \theta(D_{q,z})\exp_q(xz) \tag{4.13}$$

and can therefore be thought of as being obtained as a bispectral Darboux transformation (in the sense of [4] and [13]) of the constant coefficients (in z) q-difference operator $\theta(D_{q,z})$.

Note that, if in the proof of theorem 4.2 we make in (4.11) the special choice

$$f_0(x) = \frac{\tau(qx,t) - \tau(q^{-1}x,t)}{x(q-1)}$$

leading to $\theta(x) = \tau(x,t)\tau(qx,t)$, we obtain from (4.12) and (4.13) that

$$P_b^{-1}B\tilde{\Psi} = \tau(qx,t)\exp_q(xz). \tag{4.14}$$

Recalling that $D_q^N = QP$, with Q, P as in (3.13) and (3.14), we deduce that

$$\begin{aligned} P_b^{-1}B(z^{N-n}\exp_q(xz)) &= P_b^{-1}BQ\tilde{\Psi} \\ &= Q\tau(qx,t)\exp_q(xz). \end{aligned}$$

This last formula and formula (4.14) show that, for the special choice $\theta(x) = \tau(x,t)\tau(qx,t)$, the pseudo-difference operator $Q_b = P_b^{-1}B$ becomes a q-difference operator and equation (4.12) is nicely completed with

$$\exp_q(xz) = \frac{1}{z^{N-n}}Q\tilde{\Psi} = \frac{1}{\tau(qx,t)}Q_b\tilde{\Psi} \tag{4.15}$$

providing us with a beautiful example of a monomial bispectral Darboux transformation in the sense of [4], in the context of q-difference operators.

Example. In order to illustrate theorem 4.2, we give some explicit formulae for the q-analogue of the simplest non-trivial KdV example discussed in [7], which corresponds to $\lambda = \{\lambda_1 = 2 > \lambda_2 = 1 > 0\}$. The corresponding tau-function $\tau(x,t) = S_{2,1}(x;t_1,t_2,t_3)$ is written in (3.4). We shall denote by

$$[\alpha] = \frac{q^\alpha - 1}{q-1}$$

the q-analogue of α. In this example, we factorize D_q^4 as

$$D_q^4 = [Q\tau(qx,t)]\frac{1}{\tau(qx,t)\tau(x,t)}[\tau(x,t)P]$$

with

$$\tau(x,t)P = \tau(x,t)D_q^2 - (x+t_1)^2 D_q + (x+t_1)$$

and

$$Q\tau(qx,t) = D_q^2\tau(qx,t) + (q^2x+t_1)^2 D_q + q[3]_q x + (2q+1)t_1.$$

This gives

$$\tau(x,t)P(\exp_q(xz)) = z^2 P_b(\exp_q(xz))$$

with

$$P_b = \frac{q}{[3]_q}D_{q,z}^3 + \left(t_1 - \frac{1}{z}\right)D_{q,z}^2 + \left(t_1 - \frac{1}{z}\right)^2 D_{q,z} + \left(\frac{t_1^3}{3} - t_3 - \frac{t_1^2}{z} + \frac{t_1}{z^2}\right)$$

and

$$Q(\tau(qx,t)\exp_q(xz)) = Q_b(z^2\exp_q(xz))$$

with

$$Q_b = \frac{q^4}{[3]_q}D_{q,z}^3 + \left(t_1q^2 + \frac{1}{z}\right)D_{q,z}^2 + \left(t_1^2 q + \frac{2t_1}{z} - \frac{1}{z^2}\right)D_{q,z} + \left(\frac{t_1^3}{3} - t_3 + \frac{t_1^2}{z} - \frac{t_1}{z^2}\right).$$

This leads us from (4.12) and (4.15) to a bispectral operator $B_6 = P_bQ_b$ of order 6 with $\theta(x) = \tau(x,t)\tau(qx,t)$. By picking $f_0 = 1$ in (4.11) we can also obtain a bispectral operator B_4 of order 4 such that, according to (4.13),

$$B_4 = P_b\theta(D_{q,z})P_b^{-1}$$

with

$$\theta(x) = \left(\int \tau(x,t)\, \mathrm{d}_q x \right)_{|xq}$$
$$= \frac{q^5 x^4}{[3][4]} + t_1 \frac{q^3 x^3}{[3]} + t_1^2 \frac{q^2 x^2}{[2]} + \left(\frac{t_1^3}{3} - t_3 \right) q x$$

which is the exact q-analogue of the bispectral operator computed in [7].

Acknowledgments

LH thanks F A Grünbaum for his collaboration in [10, 11] which aroused his interest in this subject. We also thank E Horozov for explaining to us parts of his recent joint work with B Bakalov and M Yakimov. PI acknowledges the support of a FDS grant at the University of Louvain.

References

[1] Adler M and Moser J 1978 On a class of polynomials connected with the Korteweg–de Vries equation *Commun. Math. Phys.* **61** 1–30
[2] Andrews G E and Askey R 1985 Classical orthogonal polynomials *Polynômes Orthogonaux et Applications (Lecture Notes in Mathematics 1171)* ed C Brezinski *et al* (New York: Springer) pp 36–62
[3] Askey R and Wilson J 1985 Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials *Mem. Am. Math. Soc.* **319**
[4] Bakalov B, Horozov E and Yakimov M 1997 Bispectral algebras of commuting ordinary differential operators *Preprint* q-alg/9602011 to appear in *Commun. Math. Phys.*
[5] Bakalov B, Horozov E and Yakimov M 1996 General methods for constructing bispectral operators *Phys. Lett.* A **222** 59–66
[6] Dickey L A 1991 *Soliton Equations and Hamiltonian Systems (Adv. Ser. Math. Phys. 12)* (River Edge, NJ: World Scientific)
[7] Duistermaat J J and Grünbaum F A 1986 Differential equations in the spectral parameter *Commun. Math. Phys.* **103** 177–240
[8] Frenkel E 1996 Deformations of the KdV hierarchy and related soliton equations *IMRN (Internat. Mat. Res. Notices)* **2** 55–76
[9] Frenkel E and Reshetikhin N 1996 Quantum affine algebras and deformations of the Virasoro and W-algebras *Commun. Math. Phys.* **178** 237–64
[10] Grünbaum F A and Haine L 1996 The q-version of a theorem of Bochner *J. Comput. Appl. Math.* **68** 103–14
[11] Grünbaum F A and Haine L 1997 Some functions that generalize the Askey–Wilson polynomials *Commun. Math. Phys.* **184** 173–202
[12] Grünbaum F A and Haine L 1997 Bispectral Darboux transformations: an extension of the Krall polynomials *IMRN (Internat. Mat. Res. Notices)* **8** 359–92
[13] Kasman A and Rothstein M 1997 Bispectral Darboux transformations: the generalized Airy case *Physica* **102D** 159–76
[14] Reach M 1988 Difference equations for N-soliton solutions to KdV *Phys. Lett.* **129A** 101–5
[15] Sato M and Sato Y 1982 Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds *Lect. Notes Num. Appl. Anal.* **5** 259–71
[16] Segal G and Wilson G 1985 Loop groups and equations of KdV type *Publ. Math. IHES* **61** 5–65
[17] Wilson G 1993 Bispectral commutative ordinary differential operators *J. Reine Angew. Math.* **442** 177–204
[18] Zubelli J P 1992 On the polynomial τ-functions for the KP hierarchy and the bispectral property *Lett. Math. Phys.* **24** 41–8